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# Uniqueness of a pre-generator for $C_0$ -semigroup on a general locally convex vector space\*

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## Abstract

The main purpose is to generalize a theorem of Arendt about uniqueness of  $C_0$ -semigroups from Banach space setting to the general locally convex vector spaces, more precisely, we show that cores are the only domains of uniqueness for  $C_0$ -semigroups on locally convex spaces. As an application, we find a necessary and sufficient condition for that the mass transport equation has one unique  $L^1(\mathbb{R}^d, dx)$  weak solution.

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**Running head:** Uniqueness of a pre-generator

## 1 Framework and main result

The theory of  $C_0$ -semigroups of linear operators in Banach spaces was extended by SCHWARTZ [Sch'58], MIYADERA [Mi'59], YOSIDA [Yo'71], KOMATSU [Ko'64], KŌMURA [Kō'68] and others to the case of equicontinuous  $C_0$ -semigroups of linear operators in locally convex spaces. Let  $(\mathcal{X}, \beta)$  be a locally convex Hausdorff space. Recall that a family  $\{T(t)\}_{t \geq 0}$  of linear continuous operators on  $\mathcal{X}$  is called a  $C_0$ -semigroup on  $\mathcal{X}$  if the following properties holds:

- (i)  $T(0) = I$ ;
  - (ii)  $T(t + s) = T(t)T(s)$ , for all  $t, s \geq 0$ ;
  - (iii)  $\lim_{t \searrow 0} T(t)x = x$ , for all  $x \in \mathcal{X}$ ;
  - (iv) there exist a number  $\omega \in \mathbb{R}$  such that the family  $\{e^{-\omega t}T(t)\}_{t \geq 0}$  is equicontinuous.
- Furthermore we say that  $\{T(t)\}_{t \geq 0}$  is an equicontinuous  $C_0$ -semigroup if  $\omega = 0$  in (iv). The equicontinuity must be considered in the sense of seminorms: a family  $\mathcal{F}$  of linear operators on  $\mathcal{X}$  is said to be equicontinuous if for each continuous seminorm  $p$  on  $\mathcal{X}$ , there is a continuous seminorm  $q$  on  $\mathcal{X}$  such that

$$p(Tx) \leq q(x) \quad , \quad \forall T \in \mathcal{F} \text{ and } \forall x \in \mathcal{X}.$$

The *infinitesimal generator* of  $C_0$ -semigroup  $\{T(t)\}_{t \geq 0}$  is a linear operator  $\mathcal{L}$  defined on the domain

$$\mathcal{D}(\mathcal{L}) = \left\{ x \in \mathcal{X} \mid \lim_{t \searrow 0} \frac{T(t)x - x}{t} \text{ exists in } (\mathcal{X}, \beta) \right\}$$

by

$$\mathcal{L}x = \lim_{t \searrow 0} \frac{T(t)x - x}{t}, \quad \forall x \in \mathcal{D}(\mathcal{L}).$$

If the locally convex Hausdorff space  $(\mathcal{X}, \beta)$  is assumed to be sequentially complete, then:

- (i)  $\mathcal{L}$  is a densely defined and closed operator;
- (ii) the resolvent  $R(\lambda; \mathcal{L}) = (\lambda I - \mathcal{L})^{-1}$ , for any  $\lambda \in \rho(\mathcal{L})$  (the resolvent set of  $\mathcal{L}$ ) is well defined, continuous on  $\mathcal{X}$  and satisfies the equality

$$R(\lambda; \mathcal{L}) = \int_0^\infty e^{-\lambda t} T(t)x \, dt, \quad \forall \lambda > \omega \text{ and } \forall x \in \mathcal{X}.$$

Let  $\mathcal{A} : \mathcal{X} \longrightarrow \mathcal{X}$  be a linear operator with domain  $\mathcal{D}$  dense in  $\mathcal{X}$ .  $\mathcal{A}$  is said to be a *pre-generator*, if there exists some  $C_0$ -semigroup on  $\mathcal{X}$  such that its generator  $\mathcal{L}$  extends  $\mathcal{A}$ . We say that  $\mathcal{A}$  is an *essential generator* in  $\mathcal{X}$  (or  $\mathcal{X}$ -unique), if  $\mathcal{A}$  is closable and its closure  $\overline{\mathcal{A}}$  with respect to  $\beta$  is the generator of some  $C_0$ -semigroup on  $\mathcal{X}$ .

In general, for a  $C_0$ -semigroup  $\{T(t)\}_{t \geq 0}$  on  $(\mathcal{X}, \beta)$ , its adjoint semigroup  $\{T^*(t)\}_{t \geq 0}$  is no longer strongly continuous on the dual topological space  $\mathcal{Y}$  of  $(\mathcal{X}, \beta)$  with respect to the strong topology  $\beta(\mathcal{Y}, \mathcal{X})$  of  $\mathcal{Y}$ . In [WZ'06, p.563] ZHANG and the second named author introduced a new topology on  $\mathcal{Y}$  for which the usual semigroups in the literature becomes  $C_0$ -semigroups. That is *the topology of uniform convergence on compact subsets of  $(\mathcal{X}, \beta)$* , denoted by  $\mathcal{C}(\mathcal{Y}, \mathcal{X})$ . If moreover,  $(\mathcal{X}, \beta)$  is assumed to be quasi-complete (i.e., the bounded and closed subsets of  $(\mathcal{X}, \beta)$  are complete) then  $(\mathcal{Y}, \mathcal{C}(\mathcal{Y}, \mathcal{X}))^* = (\mathcal{X}, \beta)$  and if  $\{T(t)\}_{t \geq 0}$  is a  $C_0$ -semigroup on  $(\mathcal{X}, \beta)$  with generator  $\mathcal{L}$ , then  $\{T^*(t)\}_{t \geq 0}$  is a  $C_0$ -semigroup on  $(\mathcal{Y}, \mathcal{C}(\mathcal{Y}, \mathcal{X}))$  with generator  $\mathcal{L}^*$ .

The main purpose of this paper is to furnish a proof for the difficult implication of a theorem of WU and ZHANG [WZ'06, Theorem 2.1, p. 570] concerning uniqueness

of pre-generators on locally convex spaces (first time formulated in [Wu'98, Remarks (2.v), p. 292]).

**Theorem 1.1.** *Let  $(\mathcal{X}, \beta)$  be a locally convex Hausdorff sequentially complete space and  $\mathcal{A}$  a linear operator on  $\mathcal{X}$  with domain  $\mathcal{D}$  (the test-function space) which is assumed to be dense in  $(\mathcal{X}, \beta)$ . Assume that there is a  $C_0$ -semigroup  $\{T(t)\}_{t \geq 0}$  on  $(\mathcal{X}, \beta)$  such that its generator  $\mathcal{L}$  is an extension of  $\mathcal{A}$  (i.e.,  $\mathcal{A}$  is a pre-generator in  $(\mathcal{X}, \beta)$ ). The following assertions are equivalents:*

- (i)  $\mathcal{A}$  is a  $\mathcal{X}$ -essential generator (or  $\mathcal{X}$ -unique);
- (ii) the closure of  $\mathcal{A}$  in  $(\mathcal{X}, \beta)$  is exactly  $\mathcal{L}$  (i.e.,  $\mathcal{D}$  is a core for  $\mathcal{L}$ );
- (iii)  $\mathcal{A}^* = \mathcal{L}^*$  which is the generator of the dual  $C_0$ -semigroup  $\{T^*(t)\}_{t \geq 0}$  on  $(\mathcal{Y}, \mathcal{C}(\mathcal{Y}, \mathcal{X}))$ ;
- (iv) for some  $\lambda > \omega$  ( $\omega \in \mathbb{R}$  is the constant in definition of  $C_0$ -semigroup  $\{T(t)\}_{t \geq 0}$ ), the range  $(\lambda I - \mathcal{A})(\mathcal{D})$  is dense in  $(\mathcal{X}, \beta)$ ;
- (v) (Liouville property) for some  $\lambda > \omega$ ,  $\text{Ker}(\lambda I - \mathcal{A}^*) = \{0\}$  (i.e., if  $y \in \mathcal{D}(\mathcal{A}^*)$  satisfies  $(\lambda I - \mathcal{A}^*)y = 0$ , then  $y = 0$ );
- (vi) (uniqueness of solutions for the resolvent equation) for all  $\lambda > \omega$  and all  $y \in \mathcal{Y}$ , the resolvent equation of  $\mathcal{A}^*$

$$(\lambda I - \mathcal{A}^*)z = y$$

has the unique solution  $z = ((\lambda I - \mathcal{L})^{-1})^* y = (\lambda I - \mathcal{L}^*)^{-1} y$ ;

- (vii) (uniqueness of strong solutions for the Cauchy problem) for each  $x \in \mathcal{D}(\overline{\mathcal{A}})$ , there is a  $(\mathcal{X}, \beta)$ -unique strong solution  $v(t) = T(t)x$  of the Cauchy problem (or the Kolmogorov backward equation)

$$\begin{cases} \partial_t v(t) = \overline{\mathcal{A}}v(t) \\ v(0) = x \end{cases}$$

i.e.,  $t \mapsto v(t)$  is differentiable from  $\mathbb{R}^+$  to  $(\mathcal{X}, \beta)$  and its derivative  $\partial_t v(t)$  coincides with

$\overline{\mathcal{A}}v(t);$

(viii) (uniqueness of weak solutions for the dual Cauchy problem) for every  $y \in \mathcal{Y}$ , the dual Cauchy problem (or the Kolmogorov forward equation)

$$\begin{cases} \partial_t u(t) = \mathcal{A}^* u(t) \\ u(0) = y \end{cases}$$

has a  $(\mathcal{Y}, \mathcal{C}(\mathcal{Y}, \mathcal{X}))$ -unique weak solution  $u(t) = T^*(t)y$ . More precisely, there is a unique function  $\mathbb{R}^+ \ni t \mapsto u(t) = T^*(t)y$  which is continuous from  $\mathbb{R}^+$  to  $(\mathcal{Y}, \mathcal{C}(\mathcal{Y}, \mathcal{X}))$  such that

$$\langle x, u(t) - y \rangle = \int_0^t \langle \mathcal{A}x, u(s) \rangle ds \quad , \quad \forall x \in \mathcal{D};$$

(ix) there is only one  $C_0$ -semigroup on  $\mathcal{X}$  such that its generator extends  $\mathcal{A}$ .

Many equivalence relations above, especially the equivalence between (i), (vii), (viii) and (ix), are fundamental and well studied in the Banach space setting, see ARENDT [Ar'86], EBERLE [Eb'97], DJELLOUT [Dj'97] and the second named author [Wu'98] and [Wu'99], etc. In the local convex space framework, the equivalences between (i)-(viii) are proved in ZHANG and the second named author [WZ'06].

The main purpose of this paper is to prove the equivalence between (i) and (ix). This equivalence in the Banach space setting is the well known Arendt theorem. The implication (i) $\Rightarrow$ (ix) is immediate. Indeed, if  $\mathcal{A}$  is an essential generator in  $\mathcal{X}$  and if we suppose that  $\mathcal{L}$  and  $\mathcal{L}'$  are generators of some  $C_0$ -semigroups which extends  $\mathcal{A}$ , then by equivalence of (i) $\Leftrightarrow$ (ii), we have  $\mathcal{L} = \overline{\mathcal{A}} = \mathcal{L}'$ . It follows that there is only one  $C_0$ -semigroup on  $\mathcal{X}$  such that its generator extends  $\mathcal{A}$ .

The sufficiency of (ix) is difficult. We shall follow the strategy of Arendt in the Banach space setting, but several basic ingredients require much more difficult proofs in

the actual locally convex vector space setting. The main idea for overcoming those difficulties is to use the notion of calibration and to choose a "good" calibration.

This paper is organized as follows. In the next section we show that cores are the only domains of uniqueness for  $C_0$ -semigroups on locally convex spaces and we prove the sufficiency of (ix) in Theorem 1.1. Finally is presented the  $L^1$ -uniqueness for weak solution of mass transport equation.

## 2 Domains of uniqueness

Recall at first several well known facts for calibration. A *calibration* for a locally convex space  $(\mathcal{X}, \beta)$  is a family  $\Gamma$  of continuous seminorms which induces the topology  $\beta$  of  $\mathcal{X}$ . Such a family of seminorms was used by FATTORINI [Fa'68], MOORE [Mo'69], CHILANA [Ch'70], CHOE [Ch'85] and others.

Let  $p \in \Gamma$ . A linear operator  $T$  on  $\mathcal{X}$  is said to be *p-continuous* if

$$\tilde{p}(T) := \sup_{p(x) \leq 1} p(Tx) < \infty$$

and is said to be  $\Gamma$ -continuous if it is  $p$ -continuous for every  $p \in \Gamma$ . We say that  $T$  is  $\Gamma$ -bounded if

$$\|T\|_\Gamma := \sup_{p \in \Gamma} \tilde{p}(T) < \infty \quad .$$

If  $\|T\|_\Gamma \leq 1$ , then we say that  $T$  is a  $\Gamma$ -contraction.

The following result obtained by MOORE [Mo'69, Theorem 4, p. 70], give a very nice characterisation of equicontinuous semigroups.

**Lemma 2.1.** *A semigroup  $\mathcal{F}$  of linear operators on  $\mathcal{X}$  is equicontinuous if and only if there is a calibration  $\Gamma$  for  $\mathcal{X}$  such that  $\mathcal{F}$  is a semigroup of  $\Gamma$ -contraction.*

Finally, the following perturbation result of CHOE [Ch'85, Corollary 5.4, p. 312], plays a key role in the proof of our next theorem.

**Lemma 2.2.** *Let  $\Gamma$  be a calibration for a locally convex space  $(\mathcal{X}, \beta)$ . If  $A$  is the generator of a  $C_0$ -semigroup on  $\mathcal{X}$  and  $B$  is a  $\Gamma$ -bounded linear operator on  $\mathcal{X}$ , then  $A + B$  is the generator of a  $C_0$ -semigroup on  $\mathcal{X}$ .*

We turn now to the job. We begin with the following theorem which is well known in the Banach space setting (see [Ar'86, Theorem 1.33, p. 46]).

**Theorem 2.3.** *Let  $(\mathcal{X}, \beta)$  be a locally convex Hausdorff sequentially complete space,  $\{T(t)\}_{t \geq 0}$  a  $C_0$ -semigroup on  $\mathcal{X}$  with generator  $\mathcal{L}$  and  $\mathcal{D}$  a subspace of  $\mathcal{D}(\mathcal{L})$ . Consider the restriction  $\mathcal{A}$  of  $\mathcal{L}$  to  $\mathcal{D}$ . If  $\mathcal{D}$  is not a core of  $\mathcal{L}$ , then there exists an infinite number of extensions of  $\mathcal{A}$  which are generators.*

**Proof. Step 1.** Endow  $\mathcal{D}(\mathcal{L})$  with the graph topology  $\beta_{\mathcal{L}}$  of  $\mathcal{L}$  induced by the  $\beta$ -topology. If in contrary  $\mathcal{D}$  is not a core of  $\mathcal{L}$ , then  $\mathcal{D}$  is not dense in  $\mathcal{D}(\mathcal{L})$  with respect to the graph topology  $\beta_{\mathcal{L}}$ . By Hahn-Banach theorem there exist some non-zero linear functional  $\phi$  continuous on  $\mathcal{D}(\mathcal{L})$  with respect to the graph topology  $\beta_{\mathcal{L}}$  such that  $\phi(x) = 0$  for all  $x \in \mathcal{D}$ . Fix some  $u \in \mathcal{D}(\mathcal{L})$ ,  $u \neq 0$ , we consider the linear operator

$$C : \mathcal{D}(\mathcal{L}) \longrightarrow \mathcal{D}(\mathcal{L})$$

$$Cx = \phi(x)u \quad , \quad \forall x \in \mathcal{D}(\mathcal{L}).$$

Then  $C$  is  $\beta_{\mathcal{L}}$ -continuous (i.e. continuous with respect to the graph topology  $\beta_{\mathcal{L}}$ ) on  $\mathcal{D}(\mathcal{L})$ . Notice that  $C$  is  $\beta_{\mathcal{L}}$ -continuous iff for some (or all)  $\lambda_0 \in \rho(\mathcal{L})$

$$\tilde{C} := (\lambda_0 I - \mathcal{L})CR(\lambda_0; \mathcal{L})$$



is  $\beta$ -continuous on  $\mathcal{X}$ .

Let  $\Theta = CR(\lambda_0; \mathcal{L})$ . Since for all  $x \in \mathcal{X}$  we have

$$\Theta x = CR(\lambda_0; \mathcal{L})x = \phi(R(\lambda_0; \mathcal{L})x)u$$

$$\begin{aligned}\Theta^2 x &= \Theta(\Theta x) = \phi(R(\lambda_0; \mathcal{L})\Theta x)u = \phi(R(\lambda_0; \mathcal{L})\phi(R(\lambda_0; \mathcal{L})x)u)u = \\ &= \phi(R(\lambda_0; \mathcal{L})x)\phi(R(\lambda_0; \mathcal{L})u)u,\end{aligned}$$

and successively

$$\Theta^n x = \phi(R(\lambda_0; \mathcal{L})x)\phi^{n-1}(R(\lambda_0; \mathcal{L})u)u$$

for all  $n \in \mathbb{N}^*$ . One can take  $u \in \mathcal{D}(\mathcal{L})$ ,  $u \neq 0$  such that

$$|\phi(R(\lambda_0; \mathcal{L})u)| < 1.$$

Therefore the linear operator  $U = I - CR(\lambda_0; \mathcal{L})$  is invertible and both  $U$  and  $U^{-1}$ ,

$$U^{-1}x = \sum_{n=0}^{\infty} \Theta^n x = x + \phi(R(\lambda_0; \mathcal{L})x) \frac{1}{1 - \phi(R(\lambda_0; \mathcal{L})u)}u$$

are  $\beta$ -continuous on  $\mathcal{X}$ . Moreover, as in the proof of [Ar'86, Theorem 1.31, p. 45], we have

$$\begin{aligned}U(\mathcal{L} + \tilde{C})U^{-1} &= U(\mathcal{L} - \lambda_0 I + \lambda_0 I + \tilde{C})U^{-1} = \\ &= U(\mathcal{L} - \lambda_0 I + \tilde{C})U^{-1} + \lambda_0 I = \\ &= U(\mathcal{L} - \lambda_0 I + (\lambda_0 I - \mathcal{L})CR(\lambda_0; \mathcal{L}))U^{-1} + \lambda_0 I = \\ &= U(\mathcal{L} - \lambda_0 I)(I - CR(\lambda_0; \mathcal{L}))U^{-1} + \lambda_0 I = \\ &= U(\mathcal{L} - \lambda_0 I) + \lambda_0 I = \\ &= [I - CR(\lambda_0; \mathcal{L})](\mathcal{L} - \lambda_0 I) + \lambda_0 I = \\ &= \mathcal{L} - \lambda_0 I + C + \lambda_0 I = \mathcal{L} + C \quad .\end{aligned}$$

Now we have only to prove that  $\mathcal{L} + \tilde{C}$  is the generator of some  $C_0$ -semigroup on  $(\mathcal{X}, \beta)$ .

**Step 2.** To apply Lemma 2.2 of Choe, we have to find a good calibration, which is the main difficult point. Since  $\{T(t)\}_{t \geq 0}$  is a  $C_0$ -semigroup on  $\mathcal{X}$ , there exists a number  $\omega \in \mathbb{R}$  such that  $\{e^{-\omega t} T(t)\}_{t \geq 0}$  is equicontinuous. According to Lemma 2.1, there is a calibration  $\Gamma$  for  $(\mathcal{X}, \beta)$  such that

$$\|e^{-\omega t} T(t)\|_{\Gamma} \leq 1 \quad , \quad \forall t \geq 0.$$

For each  $p \in \Gamma$  we define

$$\hat{p}(x) = \sup_{t \geq 0} [p(e^{-\omega t} T(t)x) + |\phi(R(\lambda_0; \mathcal{L})e^{-\omega t} T(t)x)|] \quad , \quad \forall x \in \mathcal{X}.$$

As  $\hat{p} \geq p$  and  $\hat{p}$  is continuous, the family  $\hat{\Gamma} = \{\hat{p} \mid p \in \Gamma\}$  is another calibration of  $(\mathcal{X}, \beta)$ , which will be our calibration. We consider now the  $\hat{\Gamma}$ -norm

$$\|\tilde{C}\|_{\hat{\Gamma}} = \sup_{\hat{p} \in \hat{\Gamma}} \sup_{\hat{p}(x) \leq 1} \hat{p}(\tilde{C}x)$$

and we prove that  $\tilde{C}$  is  $\hat{\Gamma}$ -bounded, i.e.

$$\|\tilde{C}\|_{\hat{\Gamma}} < \infty \quad .$$

Let  $x \in \mathcal{X}$ . Then we have

$$\begin{aligned} \tilde{C}x &= (\lambda_0 I - \mathcal{L}) C R(\lambda_0; \mathcal{L})x = (\lambda_0 I - \mathcal{L}) \phi(R(\lambda_0; \mathcal{L})x) u = \\ &= \phi(R(\lambda_0; \mathcal{L})x) (\lambda_0 I - \mathcal{L}) u = \phi(R(\lambda_0; \mathcal{L})x) v \end{aligned}$$

where we denote  $(\lambda_0 I - \mathcal{L}) u = v$ . Therefore

$$\begin{aligned} \hat{p}(\tilde{C}x) &= \hat{p}(\phi(R(\lambda_0; \mathcal{L})x) v) = |\phi(R(\lambda_0; \mathcal{L})x)| \hat{p}(v) \\ &\leq [p(x) + |\phi(R(\lambda_0; \mathcal{L})x)|] \hat{p}(v) \leq \hat{p}(x) \hat{p}(v) \quad . \end{aligned}$$

Consequently

$$\sup_{\hat{p}(x) \leq 1} \hat{p}(\tilde{C}x) \leq \sup_{\hat{p}(x) \leq 1} \hat{p}(x)\hat{p}(v) \leq \hat{p}(v) \quad .$$

Then  $\|\tilde{C}\|_{\hat{\Gamma}} < \hat{p}(v)$ , i.e.  $\tilde{C}$  is  $\hat{\Gamma}$ -bounded. So by Lemma 2.2  $\mathcal{L} + \tilde{C}$  generate a  $C_0$ -semigroup  $\{\tilde{S}(t)\}_{t \geq 0}$ . Consequently

$$S(t) = U\tilde{S}(t)U^{-1}$$

is a  $C_0$ -semigroup whose generator is  $\mathcal{L} + C$  and  $\mathcal{L} + C/\mathcal{D} = \mathcal{L}/\mathcal{D}$ . As the choice of  $u$  above is infinite, we have proved the result. ■

**Proof of Theorem 1.1 (ix)  $\Rightarrow$  (i)** Suppose that there is only one  $C_0$ -semigroup on  $\mathcal{X}$  such that its generator extends  $\mathcal{A}$ . By the Theorem 2.3 it follows that  $\mathcal{D}$  is a core of  $\mathcal{L}$ . Therefore  $\overline{\mathcal{L}/\mathcal{D}} = \mathcal{L}$ . But  $\mathcal{A} = \mathcal{L}/\mathcal{D}$ , we conclude that  $\mathcal{A}$  is a  $\mathcal{X}$ -unique. ■

**Remarque 2.4.** If  $\mathcal{A}$  is a second order elliptic differential operator with  $\mathcal{D} = C_0^\infty(D)$ , then the weak solutions for the dual Cauchy problem in the Theorem 1.1 (viii) correspond exactly to those in the distribution sense in the theory of partial differential equations and the dual Cauchy problem becomes the Fokker-Planck equation for heat diffusion. We must remarks the important equivalences between the  $\mathcal{X}$ -uniqueness of the linear operator  $\mathcal{A}$ , the  $\mathcal{X}$ -uniqueness of strong solutions for the Cauchy problem and the  $\mathcal{Y}$ -uniqueness of weak solutions for the dual Cauchy problem associated with  $\mathcal{A}$ .

### 3 $L^1(\mathbb{R}^d, dx)$ -uniqueness of weak solutions for the mass transport equation

Consider the operator

$$\mathcal{A}f = b\nabla f \quad , \quad \forall f \in C_0^\infty(\mathbb{R}^d)$$

where the vector field  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is locally Lipschitz. Let  $\partial$  be the point at infinity of  $\mathbb{R}^d$ . Consider the ordinary differential equation (ODE)

$$\begin{cases} dX_t = b(X_t)dt \\ X(0) = x \end{cases} .$$

For every  $x \in \mathbb{R}^d$ , there is a unique solution  $(X_t(x))_{0 \leq t < \tau_e}$ , where

$$\tau_e = \inf \{t \geq 0 \mid X_t = \partial\}$$

is the explosion time. Then the family  $\{P_t\}_{t \geq 0}$ , where

$$P_t f(x) = f(X_t(x))1_{[t < \tau_e]}$$

is a  $C_0$ -semigroup on  $L^\infty(\mathbb{R}^d, dx)$  with respect to the topology  $\mathcal{C}(L^\infty, L^1)$  and

$$f(X_t) - f(X_0) = \int_0^t b\nabla f(X_s)ds \quad , \quad \forall f \in C_0^\infty(\mathbb{R}^d).$$

Therefore  $f$  belongs to the domain of the generator  $\mathcal{L}_{(\infty)}$  of  $C_0$ -semigroup  $\{P_t\}_{t \geq 0}$  on  $L^\infty(\mathbb{R}^d, dx)$  and

$$\mathcal{L}_{(\infty)}f = \mathcal{A}f = b\nabla f \quad .$$

Consequently,  $(\mathcal{A}, C_0^\infty(\mathbb{R}^d))$  is a pre-generator on  $(L^\infty(\mathbb{R}^d, dx), \mathcal{C}(L^\infty, L^1))$ . So we can study the  $(L^\infty(\mathbb{R}^d, dx), \mathcal{C}(L^\infty, L^1))$ -uniqueness of the operator  $(\mathcal{A}, C_0^\infty(\mathbb{R}^d))$ .

Consider at first the one-dimensional operator

$$\mathcal{A}f = bf' \quad , \quad \forall f \in C_0^\infty(\mathbb{R})$$

where  $b$  is a locally Lipschitz continuous function on  $\mathbb{R}$  such that  $b(x) > 0$ , for all  $x \in \mathbb{R}$ . Let  $\mathcal{A}^* : \mathcal{D}(\mathcal{A}^*) \subset L^1(\mathbb{R}, dx) \rightarrow L^1(\mathbb{R}, dx)$  the adjoint operator of  $\mathcal{A}$  and  $h \in L^1(\mathbb{R}, dx)$  such that  $h \in \mathcal{D}(\mathcal{A}^*)$  and

$$\mathcal{A}^*h = \lambda h \quad .$$

Then  $h$  solve the ODE in the distribution sense

$$-(bh)' = \lambda h \quad .$$

Then  $bh$  is absolutely continuous where it follows that  $h$  is absolutely continuous in the set  $\{x \in \mathbb{R} \mid b(x) \neq 0\}$ .

**Theorem 3.1.** *Assume that  $b$  is locally Lipschitzian and  $b(x) > 0$  over  $\mathbb{R}$ . Then  $(\mathcal{A}, C_0^\infty(\mathbb{R}))$  is  $(L^\infty(\mathbb{R}, dx), \mathcal{C}(L^\infty, L^1))$ -unique if and only if*

$$\int_{-\infty}^0 \frac{1}{b(x)} dx = \infty \quad .$$

**Proof.** As shown before  $(\mathcal{A}, C_0^\infty(\mathbb{R}))$  is a pre-generator on  $(L^\infty(\mathbb{R}, dx), \mathcal{C}(L^\infty, L^1))$ .

*Sufficiency.* Suppose in contrary that  $(\mathcal{A}, C_0^\infty(\mathbb{R}))$  is not  $(L^\infty(\mathbb{R}, dx), \mathcal{C}(L^\infty, L^1))$ -unique. Then there is a function  $h \in L^1(\mathbb{R}, dx)$ ,  $h \neq 0$  such that

$$(I - \mathcal{A}^*)h = 0$$

in the sense of distributions. Thus we may assume that  $h$  is itself absolutely continuous and solves the ODE

$$-(bh)' = h \quad .$$

Then

$$h' = -\frac{1+b'}{b}h \quad ,$$

we find

$$h(x) = h(0)e^{-\int_0^x \frac{1+b'(s)}{b(s)} ds} = h(0)\frac{b(0)}{b(x)}e^{-\int_0^x \frac{1}{b(s)} ds}.$$

Because  $h \neq 0$ , we have  $h(0) \neq 0$ . Then

$$\int_{\mathbb{R}} |h(x)| dx = |h(0)||b(0)| \int_{\mathbb{R}} \frac{1}{b(x)} e^{-\int_0^x \frac{1}{b(s)} ds} dx$$

and for

$$u(x) = \int_0^x \frac{1}{b(s)} ds$$

$u(-\infty) = -\infty$ , we obtain

$$\int_{\mathbb{R}} |h(x)| dx = |h(0)||b(0)| \int_{u(-\infty)}^{u(\infty)} e^{-u} du = \infty$$

which is in contradiction with the assumption that  $h \in L^1(\mathbb{R}, dx)$ .

*Necessity.* If in contrary

$$\int_{-\infty}^0 \frac{1}{b(x)} dx < \infty \quad ,$$

then

$$h(x) = \frac{b(0)}{b(x)} e^{-\int_0^x \frac{1}{b(s)} ds} \in L^1(\mathbb{R}, dx)$$

and

$$(I - \mathcal{A}^*)h = 0$$

which is contradictory to the fact that  $\mathcal{A}$  is  $(L^\infty(\mathbb{R}, dx))$ -unique. ■

We can formulate next

**Theorem 3.2.** Let  $b$  be a locally Lipschitz continuous function on  $\mathbb{R}$  such that for some  $c_0 < c_N \in \mathbb{R}$ ,  $b1_{(-\infty, c_0]}$  and  $b1_{[c_N, +\infty)}$  keep a constant sign (non-zero). Then  $(\mathcal{A}, C_0^\infty(\mathbb{R}))$  is  $(L^\infty(\mathbb{R}, dx), \mathcal{C}(L^\infty, L^1))$ -unique if and only if

$$\int_{-\infty}^{c_0} \frac{1}{b^+(x)} dx = \int_{c_N}^{+\infty} \frac{1}{b^-(x)} dx = +\infty.$$

**Proof. Sufficiency.** Suppose in contrary that  $(\mathcal{A}, C_0^\infty(\mathbb{R}))$  is not  $(L^\infty(\mathbb{R}, dx), \mathcal{C}(L^\infty, L^1))$ -unique. Then there is a function  $h \in L^1(\mathbb{R}, dx)$ ,  $h \neq 0$  such that

$$(I - \mathcal{A}^*)h = 0$$

in the sense of distributions. Then  $bh$  is absolutely continuous over  $\mathbb{R}$ .

Because  $b1_{(-\infty, c_0]}$  and  $b1_{[c_N, +\infty)}$  keep a constant sign, we may suppose that

$$\{x \in \mathbb{R} \mid b(x) = 0\} = \{x_1 < x_2 < \dots < x_N\} \subset [c_0, c_N].$$

**Step 1.** Let  $x \in I_k = (x_k, x_{k+1})$  and  $c_k \in I_k$ ,  $k \in \{1, 2, \dots, N-1\}$ . Since  $h$  is absolutely continuous over  $I_k$ , we have

$$h(x) = h(c_k) \frac{b(c_k)}{b(x)} e^{-\int_{c_k}^x \frac{1}{b(s)} ds}, \quad \forall x \in I_k.$$

Because  $h \neq 0$  in  $I_k$ , we have

- if  $b(x) > 0$  for all  $x \in I_k$ , then

$$\lim_{x \searrow x_k} b(x)h(x) = b(c_k)h(c_k)e^{\int_{c_k}^{x_k} \frac{1}{b(s)} ds} \text{ not exist}$$

- if  $b(x) < 0$  for all  $x \in I_k$ , then

$$\lim_{x \nearrow x_{k+1}} b(x)h(x) = b(c_k)h(c_k)e^{-\int_{c_k}^{x_{k+1}} \frac{1}{b(s)} ds} \text{ not exist}$$

But all this are in contradiction with ours suppositions.

**Step 2.** Let  $x \in (x_N, \infty)$ . Then

$$h(x) = h(c_N) \frac{b(c_N)}{b(x)} e^{-\int_{c_N}^x \frac{1}{b(s)} ds},$$

where we deduce that

$$\int_{x_N}^{+\infty} h(x) dx = h(c_N) b(c_N) \int_{x_N}^{+\infty} \frac{1}{b(x)} e^{-\int_{c_N}^x \frac{1}{b(s)} ds} dx.$$

Let

$$u(x) = \int_{c_N}^x \frac{1}{b(s)} ds.$$

Because  $h(c_N) \neq 0$ , we have

- if  $b(x) < 0$  over  $(x_N, +\infty)$ , then  $u(+\infty) = -\infty$  and

$$\int_{x_N}^{\infty} h(x) dx = h(c_N) b(c_N) \int_{u(x_N)}^{u(\infty)} e^{-u} du = \infty \text{ or } -\infty$$

- if  $b(x) > 0$  for all  $x \in (x_N, \infty)$ , then

$$\lim_{x \searrow x_N} b(x) h(x) = h(c_N) b(c_N) e^{\int_{c_N}^{x_N} \frac{1}{b(s)} ds} = \infty \text{ or } -\infty$$

All this are again in contradiction with ours suppositions.

**Step 3.** The case where  $x \in (-\infty, x_1)$  can be treated like the Step 2.

*Necessity.* Suppose in contrary that one of

$$\int_{-\infty}^{c_0} \frac{1}{b^+(x)} dx \quad \text{or} \quad \int_{c_N}^{+\infty} \frac{1}{b^-(x)} dx$$



is finite. We work only in the case where

$$\int_{c_N}^{+\infty} \frac{1}{b^-(x)} dx < \infty$$

and the other case can be treated in the same way. Define

$$h(x) = \begin{cases} \frac{b(c_N)}{b(x)} e^{-\int_{c_N}^x \frac{1}{b(s)} ds} & , \quad x > x_N \\ 0 & , \quad x \leq x_N \end{cases}$$

We have  $-(bh)' = h$  on  $(-\infty, x_N)$  and  $(x_N, \infty)$ . Since

$$\lim_{x \searrow x_N} b(x)h(x) = b(c_N) e^{\int_{c_N}^{x_N} \frac{1}{b(s)} ds} = 0,$$

the function  $h$  is again a solution of  $-(bh)' = h$  in the sense of distribution, which is in contradiction with the fact that  $\mathcal{A}$  is  $(L^\infty(\mathbb{R}, dx))$ -unique. ■

In the multidimensional case  $d \geq 2$ , the main result of this section is

**Theorem 3.3.** *Let  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a function of the class  $C^1(\mathbb{R}^d)$  such that  $b(x) \neq 0$  for all  $|x| \geq R$ . Suppose that there is a locally bounded function  $\beta : \mathbb{R}^+ \rightarrow \mathbb{R}$  such that*

$$\left( b(x) \frac{x}{|x|} \right)^- \leq \beta(|x|) \quad \forall |x| \geq R \quad .$$

*If*

$$\int_R^\infty \frac{1}{\beta(x)} dx = \infty \quad ,$$

*then  $(\mathcal{A}, C_0^\infty(\mathbb{R}^d))$  is  $(L^\infty(\mathbb{R}^d, dx), \mathcal{C}(L^\infty, L^1))$ -unique. In particular, for all  $h \in L^1(\mathbb{R}^d, dx)$ , the mass transport equation*

$$\begin{cases} \partial_t \rho(t, x) = -\operatorname{div}(b\rho(t, x)) \\ \rho(0, x) = h(x) \end{cases}$$

*has one  $L^1(\mathbb{R}^d, dx)$ -unique weak solution.*

**Proof.** For all  $x \in \mathbb{R}^d$ , consider  $(X_t(x))_{0 \leq t < e}$ , where  $e$  is the explosion time, the solution of the equation

$$\begin{cases} \frac{dX_t}{dt} = b(X_t) \\ X(0) = x \end{cases}.$$

Then the family  $\{P_t\}_{t \geq 0}$ ,

$$P_t(x) = f(X_t(x))$$

is a  $C_0$ -semigroup on  $L^\infty(\mathbb{R}^d, dx)$  with respect to the topology  $\mathcal{C}(L^\infty, L^1)$ .

**Step 1.** We first prove that if  $f \in C_0^1$ , there exists  $(f_n)_{n \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$  such that  $f_n \rightarrow f$  and  $\mathcal{A}f_n \rightarrow \mathcal{A}f$  in the topology  $\mathcal{C}(L^\infty, L^1)$ .

Indeed, let  $\text{supp} f \subset B(0, N)$ . Then by convolution method, there exists  $(f_n)_{n \in \mathbb{N}} \in C_0^\infty(\mathbb{R}^d)$  such that  $\text{supp} f_n \subset B(0, N+1)$ , for all  $n \geq 1$ ,  $f_n \rightarrow f$  and  $\nabla f_n \rightarrow \nabla f$  uniformly over  $\mathbb{R}^d$ . Thus

$$b \nabla f_n \rightarrow b \nabla f$$

uniformly over  $\mathbb{R}^d$ .

**Step 2.** It remains to prove that  $C_0^1$  is a core for  $\mathcal{A}$ . To that purpose, by [WZ'06, Lemma 2.4, p.572], it is enough to show that

$$P_t C_0^1 \subset C_0^1$$

or, equivalently, to establish

$$\lim_{|x| \rightarrow \infty} |X_t(x)| = \infty.$$

Consider

$$\tau_n = \inf\{t \mid |X_t| = n\}$$

and

$$\tau_\infty = \lim_{n \rightarrow \infty} \tau_n = e.$$

For all  $t < \tau_R \wedge \tau_\infty$ , we have

$$\frac{d|X_t(x)|}{dt} = \frac{X_t(x)}{|X_t(x)|} b(X_t(x)) \geq -\beta(|X_t(x)|).$$

Let

$$h(x) = \int_R^x \frac{1}{\beta(s)} ds.$$

Then we have

$$\frac{d}{dt} h(|X_t(x)|) = \frac{1}{\beta(|X_t(x)|)} \frac{X_t(x)}{|X_t(x)|} b(X_t(x)) \geq -1$$

where it follows that

$$h(|X_t(x)|) \geq h(|x|) - t \quad , \quad \forall t \in [0, \tau_R \wedge \tau_\infty).$$

Consequently

$$\lim_{|x| \rightarrow \infty} |X_t(x)| = \infty$$

where we deduce that  $(\mathcal{A}, C_0^\infty(\mathbb{R}^d))$  is  $(L^\infty(\mathbb{R}^d, dx), \mathcal{C}(L^\infty, L^1))$ -unique. ■

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